Finite convergent presentation of plactic monoid for type C

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Abstract – We give an explicit presentation for the plactic monoid for type C using admissible column generators. Thanks to the combinatorial properties of symplectic tableaux, we prove that this presentation is finite and convergent. We obtain as a corollary that plactic monoids for type C satisfy homological finiteness properties.

Keywords – Plactic monoid, crystal graphs, symplectic tableau, convergent presentations.

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1. INTRODUCTION

The plactic monoid was discovered by Knuth, [Knu70], using the tableau defined by Schensted in [Sch61] in his study of the longest increasing subsequence of a permutation. Lascoux and Schützenberger in [LS81] used it in order to give a proof of the Littlewood-Richardson rule for the decomposition of tensor products of irreducible modules on the Lie algebra of traceless square matrices.

Thanks to Kashiwara's theory of crystal bases [Kas91, KN94, Kas95], plactic monoid can be defined for all classical simple Lie algebras extending the previous definition in the case of type A. Plactic monoid for type C was described by Lascoux, Leclerc and Thibon without proof in [LLT95]. After Lecouvey in [Lec02] and Baker in [Bak00] defined it by different methods. For types B and D, the plactic monoid was introduced by Lecouvey in [Lec03].

For type A, consider the ordered alphabet $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$. For every word w over the free monoid $\mathcal{A}_n^*$, a unique tableau $\mathrm{P}(w)$ can be computed using the Schensted insertion algorithm (column insertion) [Sch61]. Let u and v be two words on $\mathcal{A}_n^*$, one can define a relation $\sim$ on the free monoid $\mathcal{A}_n^*$ by:

$$u \sim v \text{ if, and only if, } \mathrm{P}(u) = \mathrm{P}(v).$$

Then the quotient $\mathbf{Pl}(\mathbf{A}) := \mathcal{A}_n^*/\sim$ is called the plactic monoid for type A. The plactic monoid can be also described as the quotient of $\mathcal{A}_n^*$ by the Knuth relations:

$$\left\{ xzy = zxy \;\middle|\; 1 \leq x < y \leq z \leq n \right\} \cup \left\{ yxz = yzx \;\middle|\; 1 \leq x \leq y < z \leq n \right\} \tag{1}$$

which is called the *Knuth presentation*.

We deal with presentations from the rewriting theory perspective. In this context, relations are oriented and are considered as rewriting steps. A presentation terminates if it has no infinite rewriting sequence. A terminating presentation is convergent if every element admits a unique normal form. Having a finite convergent presentation of a monoid has many advantages : computation of normal form and some homological properties. An open problem was to find a finite convergent presentation of the plactic monoid.

In [CGM15], Cain, Gray and Malheiro answer positively this question. They constructed a finite presentation of the plactic monoid for type A by introducing the column generators. They proved the convergence of this presentation using the combinatorial properties of Young tableaux. But the above question was still open for the plactic monoids for the others types.

In the present work, we consider the plactic monoid for type C constructed by Lecouvey in [Lec02]. Our aim is to construct a finite convergent presentation for this monoid. The generating set of this presentation contains the finite set of admissible columns introduced by Kashiwara and Nakashima in [KN94]. As a corollary, we deduce that plactic monoids for type C satisfy some homological finiteness properties.

The paper is organised as follows. We first recall in section 2 some properties of 2-polygraph which corresponds to a presentation of a monoid by a rewriting system, that is a presentation by generators and oriented relations. After that, we present some properties of crystal graphs. In section 3 we present the definitions and some properties of admissible columns and symplectic tableaux. We describe the column insertion algorithm for type C introduced by Lecouvey in [Lec02] and a definition of the Plactic monoid for type C. In section 4, we give a finite and convergent presentation of the plactic monoid for type C using admissible column generators.

2. PRELIMINARIES

2.1. Rewriting properties of 2-polygraphs

We will give some rewriting properties of the presentations of monoids. These presentations are discussed in terms of polygraphs in [GM14]. A *2-polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ made of an oriented graph

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\Longleftarrow}} \Sigma_1$$

where Σ_0 and Σ_1 are respectively the sets of objects, or generating 0-cells and of arrows, or generating 1-cells and s_0, t_0 denote the source and target maps. The set Σ_2 is a cellular extension over the free category Σ_1^*, that is a set of 2-cells relating parallel 1-cells

$$\begin{tikzcd} x \arrow[r, bend left, "s_1(\alpha)"] \arrow[r, bend right, "t_1(\alpha)"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

such that $s_0 s_1(\alpha) = s_0 t_1(\alpha)$ and $t_0 s_1(\alpha) = t_0 t_1(\alpha)$, where $s_1(\alpha), t_1(\alpha) \in \Sigma_1^*$. In our case, the set Σ_0 contains only one 0-cell. In the sequel, a 2-polygraph is denoted by $\Sigma = (\Sigma_1, \Sigma_2)$.

A monoid M is presented by a 2-polygraph Σ if M is isomorphic to the quotient of the free monoid Σ_1^* by the congruence generated by Σ_2. Then the generating 1-cells are the generators of M and the generating 2-cells correspond to the relations of M. Note that we will also say words for the 1-cells of Σ_1^* in a case of monoid.

A 2-polygraph Σ is finite if Σ_1 and Σ_2 are finite. For two 1-cells u and v, we write $u \Rightarrow v$ for a 2-cell in Σ_2. Denote by $|w|$ the length of a word w on Σ_1^*. A *rewriting step* of Σ is a 2-cell with shape

$$\begin{tikzcd} x \arrow[r, "w"] & y \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\varphi"] & z \arrow[r, "w'"] & t \end{tikzcd}$$

where φ is a 2-cell and w and w' are 1-cells of Σ_1^*. A *rewriting sequence* of Σ is a finite or infinite sequence of rewriting steps. We say that u rewrites into v if Σ has a nonempty rewriting sequence from u to v. A 1-cell of Σ_1^* is a *normal form* if Σ has no rewriting step with source u. A normal form of u is a 1-cell v that is a normal form and such that u rewrites into v. We say that Σ *terminates* if it has no infinite rewriting sequences. We say that Σ is *confluent* if for any 1-cells u, u' and u'' of Σ_1^*, such that u rewrites into u' and u'', there exists a 1-cell v such that u' and u'' rewrite into v. We say that Σ is *convergent* if it terminates and it is confluent. Note that a terminating 2-polygraph is convergent if every 1-cell admits a unique normal form.

Two 2-polygraphs are *Tietze-equivalent* if they present the same monoid. Two finite 2-polygraphs are Tietze-equivalent if, and only if, they are related by a finite sequence of elementary Tietze transformations:

– adjunction or elimination of a 1-cell x and of a 2-cell $\alpha : u \Rightarrow x$, where u is a 1-cell of the $(\Sigma_1 \setminus \{x\})^*$,

– adjunction or elimination of a 2-cell $\alpha : u \Rightarrow v$ such that u and v are related by a nonoriented sequence of 2-cells all in $\Sigma_2 \setminus \{\alpha\}$.

2.2. Crystal graphs

The *symplectic Lie algebra* sp_{2n} is the Lie algebra of $2n$ by $2n$ matrices A, for $n > 0$, that satisfy

$$\Omega A + A^T \Omega = 0,$$

where A^T is the transpose of A and $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Let V_n be the vector representation of sp_{2n}, this representation is of dimension $2n$ and we index a basis of V_n by the set

$$\mathcal{C}_n = \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{1}\},$$

totally ordered by $1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{1}$. Denote by $\mathcal{C}_n^*$ the free monoid over $\mathcal{C}_n$.

Crystal graphs are oriented graphs with labelled arrows. An arrow $a \xrightarrow{i} b$ means that $\widetilde{f}_i(a) = b$ and $\widetilde{e}_i(b) = a$, where $\widetilde{e}_i$ and $\widetilde{f}_i$ are the crystal graph operators that we will define later in our context. Note that every representation of sp_{2n} admits a crystal graph. The crystal graph of V_n is :

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \to n-1 \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \to \overline{2} \xrightarrow{1} \overline{1}.$$

Now, we consider tensor products of the vector representations $V_n^{\otimes l}$, for any l and the infinite dimensional representation $\bigoplus\limits_{l} V_n^{\otimes l}$. The crystal graphs of these representations are denoted by $G_{n,l}$ and G_n respectively. The vertices of G_n are indexed by the words of $\mathcal{C}_n^*$ and those of $G_{n,l}$ by the words of length l.

In [KN94], Kashiwara and Nakashima describe a process to compute the action of the crystal operators $\widetilde{e}_i$ and $\widetilde{f}_i$ on a word w of $\mathcal{C}_n^*$, for a fixed i. First, one considers the subword w_i of w which contains only the letters $i, i+1, \overline{i+1}$ and $\overline{i}$. One identifies the letters i and $\overline{i+1}$ by the symbol $+$ and the letters $i+1$ and $\overline{i}$ by the symbol $-$. Secondly, we remove the subwords of length 2 in w_i which correspond to the symbol $+-$, *i.e.*, we remove adjacent letters $(i, i+1)$, $(i, \overline{i})$, $(\overline{i+1}, i+1)$ and $(\overline{i+1}, \overline{i})$. Then one obtains a new subword $w_i^{(1)}$ of w. The second step of the process is repeated until there is no possibility to remove new letters. Let r and s be respectively the number of letters corresponding to the symbol $-$ and $+$ in the final subword.

– If $r > 0$ then $\widetilde{e}_i(w)$ is obtained by replacing in w the rightmost element with the symbol $-$ of the final subword, by its corresponding element with the symbol $+$, *i.e.*, $i+1$ is transformed into i or $\overline{i}$ into $\overline{i+1}$, and the others elements of w stay unchanged. If $r = 0$, then $\widetilde{e}_i(w) = 0$.

– If $s > 0$ then $\widetilde{f}_i(w)$ is obtained by replacing in w the rightmost element with the symbol $+$ of the final subword, by its corresponding element with the symbol $-$, *i.e.*, i is transformed into $i+1$ or $\overline{i+1}$ into $\overline{i}$, and the others elements of w stay unchanged. If $s = 0$, then $\widetilde{f}_i(w) = 0$.

2.2.1. Example. Consider the word $w = \bar{3}323133\bar{3}$. Let us compute $\widetilde{e}_i(w)$ and $\widetilde{f}_i(w)$ for $i = 2$. We have

$$w_i = \bar{3}32333\bar{3} \text{ and } w_i^{(1)} = 3\bar{3}.$$

We remark that we cannot remove new letters from $w_i^{(1)}$, then $r = s = 1$ and we obtain

$$\widetilde{e}_2(w) = \bar{3}323123\bar{3} \text{ and } \widetilde{f}_2(w) = \bar{3}323133\bar{2}.$$

Thanks to the properties of crystal operators with regards to the tensor products, the crystal graph $G_{n,l}$ can be decomposed into connected components. They correspond to the crystal graphs of the irreducible representations occurring in the decomposition of $V_n^{\otimes l}$. If w is a vertex of $G_{n,l}$, the connected component of $G_{n,l}$ containing w is denoted by $B(w)$. In each connected component, there exists a unique vertex w^0 which satisfy the following property:

$$\widetilde{e}_i(w^0) = 0, \quad \text{for } i = 1, \cdots, n.$$

This vertex is called the *vertex of highest weight*, and its weight is

$$\mathrm{wt}(w^0) = d_n\Lambda_n + \sum_{i=1}^{n-1}(d_i - d_{i+1})\Lambda_i,$$

where d_i is the number of letters i in w^0 minus the number of letters $\bar{i}$, and $\Lambda_1, \cdots, \Lambda_n$ are the fundamental weights of sp_{2n}. Recall that two connected components are isomorphic if their vertex of highest weight has the same weight.

A *Young diagram* is a collection of boxes in left-justified rows, where each row has the same or shorter length than the one above it. The irreducible sp_{2n}-modules can be paramterized by Young diagrams. One associates to $\lambda = \sum_{i=1}^{n}\lambda_i\Lambda_i$ the Young diagram containing λ_i columns of height i. We say that this Young diagram has shape λ. The number of boxes of it is equal to $|\lambda| = \sum_{i=1}^{n}\lambda_i i$. Denote by $B(\lambda)$ the crystal graph of the irreducible module of highest weight λ.

2.2.2. Lemma ([KN94]). *For any words u and v on $\mathcal{C}_n^*$, the word uv is a vertex of highest weight of a connected component of G_n if, and only if, u is a vertex of highest weight and $\varepsilon_i(v) \leq \varphi_i(u)$ for any $i = 1, \cdots, n$, where*

$$\varepsilon_i(w) = \max\{k; \widetilde{e}_i^k(w) \neq 0\} \textit{ and } \varphi_i(w) = \max\{k; \widetilde{f}_i^k(w) \neq 0\},$$

for w a word on $\mathcal{C}_n^*$.

For more details about crystal graphs, the reader is referred to [Kas91, Kas95, KN94].

3. Plactic monoid for type C

3.1. Symplectic Tableaux

A *column* for type C is a Young diagram C consisting of one column filled by letters of $\mathcal{C}_n$ strictly increasing from top to bottom. Denote by $w(C)$ the word obtained by reading the letters of a column C from top to bottom. It is called the *reading* of C. The height of a column C is the number of letters in C and denoted by $h(C)$. A word w is a column word if there exists a column C such that $w = w(C)$.

For example the Young diagram

$$C = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \overline{6} \\ \hline \overline{5} \\ \hline \end{array}$$

is a column. Its reading is $w(C) = 123\overline{6}\,\overline{5}$.

In [KN94], Kashiwara and Nakashima introduce the notion of admissible column. Let $w(C) = x_1 \cdots x_{h(C)}$ be the reading of a column C. For us, the column C is *admissible* if for $m = 1, \cdots, h(C)$, the number of letters x in C such that $x \leq m$ or $x \geq \overline{m}$ is smaller or equal than m.

Let C be a column and $I = \{x_1 > \cdots > x_r\}$ be the set of unbarred letters such that the pair $(x_i, \overline{x}_i)$ occurs in C, for $i = 1, \cdots, r$. The column C can be *split* if there exists a set of unbarred letters $J = \{y_1 > \cdots > y_r\}$ which contains r elements of $\mathcal{C}_n$ such that :

- y_1 is the greatest letter of $\mathcal{C}_n$ satifying $y_1 < x_1, y_1 \notin C$, and $\overline{y_1} \notin C$,
- for $i = 2, \cdots, r$, y_i is the greatest letter of $\mathcal{C}_n$ such that $y_i < \min(y_{i-1}, x_i), y_i \notin C$ and $\overline{y_i} \notin C$.

Denote by rC the column obtained by changing in C, $\overline{x}_i$ into $\overline{y}_i$ for each letter x_i in the set I up to reordering. Denote by lC the column obtained by changing in C, x_i into y_i for each letter x_i in the set I up to reordering.

3.1.1. Proposition ([She99]). *A column* C *is admissible if, and only if, it can be split.*

3.1.2. Example. Let $w(C) = 2568\overline{8}\,\overline{5}\,\overline{2}$ be the reading of a column C. Then

$$I = \{8 > 5 > 2\}, J = \{7 > 4 > 1\}, rC = 2568\overline{7}\,\overline{4}\,\overline{1}, \text{ and } lC = 1467\overline{8}\,\overline{5}\,\overline{2}.$$

The column C can be split, then it is an admissible column.

3.1.3. Example. Let $w(C') = 2346\overline{6}\,\overline{3}\,\overline{2}$ be the reading of a column C'. Then

$$I = \{6, 3, 2\}, y_1 = 5, y_2 = 1,$$

and we cannot find an element y_3 of $\mathcal{C}_n$ such that $y_3 < 1$. Thus C' cannot be split.

Using admissible columns, one can construct a tableau whose columns are admissible with an additional property on them. This tableau is called the symplectic tableau. We will recall its definition in our context.

Let C_1 and C_2 be two admissible columns. We consider the following notation :

- $C_1 \leq C_2$ if $h(C_1) \geqslant h(C_2)$ and the rows of the tableau C_1C_2 are weakly increasing from left to right.

– $C_1 \preceq C_2$ if $rC_1 \leq lC_2$.

Consider a tableau $T = C_1 \cdots C_r$, with admissible column C_i, for $i = 1, \cdots, r$. The tableau T is a *symplectic tableau* if $C_i \preceq C_{i+1}$ for $i = 1, \cdots, r-1$. The reading of the symplectic tableau T is the word $w(T)$ obtained by reading the columns of T from right to left, that is

$$w(T) = w(C_r)w(C_{r-1}) \cdots w(C_1).$$

3.1.4. Example. Let us consider the tableau

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & \overline{3} & \\ \cline{1-2} 3 & \overline{2} & \\ \cline{1-2} \end{array}$$

T is a symplectic tableau. Indeed,

– $w(C_1) = 123$, $I_{C_1} = J_{C_1} = \varnothing$, and $rC_1 = lC_1 = 123$.

– $w(C_2) = 2\overline{3}\,\overline{2}$, $I_{C_2} = \{2\}$, $J_{C_2} = \{1\}$, $rC_2 = 2\overline{3}\,\overline{1}$, and $lC_2 = 1\overline{3}\,\overline{2}$.

– $w(C_3) = 3$, $I_{C_3} = J_{C_3} = \varnothing$, and $rC_3 = lC_3 = 3$.

The columns C_1, C_2 and C_3 can be split, then they are admissible columns. We have also $C_1 \preceq C_2 \preceq C_3$, then T is a symplectic tableau and $w(T) = 32\overline{3}\,\overline{2}123$.

Let $\lambda = \sum_{i=1}^{n} \lambda_i \Lambda_i$ be a weight with $\lambda_i \geqslant 0$. By Theorem 4.5.1 in [KN94], $B(\lambda)$ coincides with the set of symplectic tableaux of shape λ. More precisely, the readings of these tableaux are the vertices of a connected component of $G_{n,|\lambda|}$ isomorphic to $B(\lambda)$. The highest weight vertex of this component is the reading of the tableau of shape λ filled with 1 on the 1st row, 2 on the 2nd row, ... , and n on the nth row.

3.2. Definition of the plactic monoid for type C

Let u and v be two words on $\mathcal{C}_n^*$. One can define a relation $\sim$ on the free monoid $\mathcal{C}_n^*$ by : $u \sim v$ if, and only if, u and v have the same position in their isomorphic connected component $B(u)$ and $B(v)$ of the crystal G_n. In other words, there exist $i_1, \cdots, i_r$ such that $u = \widetilde{f}_{i_i} \cdots \widetilde{f}_{i_r}(u^0)$ and $v = \widetilde{f}_{i_i} \cdots \widetilde{f}_{i_r}(v^0)$.

3.2.1. Proposition ([Lec02, Proposition 3.1.2]). *Every word w on $\mathcal{C}_n^*$ admits a unique symplectic tableau T such that $w \sim w(T)$. The Tableau T is denoted by $P(w)$.*

The quotient $\mathbf{Pl}(C) := \mathcal{C}_n^*/\sim$ is called the *plactic monoid for type C* or the *symplectic plactic monoid*.

Furthermore, the plactic monoid for type C can be presented by generators and relations. Consider the congruence $\equiv$ generated by the following relations on $\mathcal{C}_n^*$:

$$(R_1) : \begin{cases} yzx \equiv yxz & \text{for } x \leq y < z \text{ with } z \neq \overline{x} \\ xzy \equiv zxy & \text{for } x < y \leq z \text{ with } z \neq \overline{x} \end{cases}$$

$$(R_2) : \begin{cases} y\overline{(x-1)}(x-1) \equiv yx\overline{x} & \text{for } 1 < x \leq n \text{ and } x \leq y \leq \overline{x} \\ x\overline{x}y \equiv \overline{(x-1)}(x-1)y & \text{for } 1 < x \leq n \text{ and } x \leq y \leq \overline{x} \end{cases}$$

(R_3) : let w be a nonadmissible column word such that each strict factor of w is an admissible column word. Let z be the lowest unbarred letter such that the pair $(z, \overline{z})$ occurs in w and $N(z) = z + 1$. Then $w \equiv \widetilde{w}$, where $\widetilde{w}$ is the column word obtained by erasing the pair $(z, \overline{z})$ in w.

3.2.2. Remark. The relations (R_1) contain the Knuth relations for type A. The relations (R_3) are called the contraction relations.

3.2.3. Theorem ([Lec02, Theorem 3.2.8]). *For any words* u *and* v *on* $\mathcal{C}_n^*$, *we have*

$$u \sim v \ \textit{if, and only if,} \ u \equiv v \ \textit{if, and only if,} \ P(u) = P(v).$$

3.3. A bumping algorithm for type C

In [Sch61], Schensted introduces an insertion algorithm (column insertion) to compute a unique tableau $P(w)$ for a word w over the alphabet $\mathcal{A}_n = \{1 < \cdots < n\}$. The column insertion procedure inserts a letter x into a tableau T as follows. Let y be the smallest element of the first column of the tableau T such that $y \geq x$. Then x remplaces y in the first column and y is bumped into the next column where the process is repeated. This procedure terminates when the letter which is bumped is greater than all the elements of the next column. Then it is placed at the bottom of that column. Hence the tableau $P(w)$ can be computed by starting with the empty word, which is a valid tableau, and iteratively applying Schensted's algorithm.

In [Lec02], Lecouvey introduces an insertion scheme to compute the symplectic tableau $P(w)$ analogous to the Schensted's algorithm for type A. The insertion of a letter x in a symplectic tableau T is denoted by $x \rightarrow T$.

3.3.1. Insertion of a letter in an admissible column. Consider a word $w = w(C)x$, where x is a letter and C is an admissible column of height p. We have three cases :

– If w is the reading of an admissible column, then $x \rightarrow C$ is the column obtained by adding a box filled by the letter x at the bottom of C. In this case, the highest weight vertex of $B(w)$ is equal to $1 \cdots p(p+1)$.

– If w is a nonadmissible column word such that each strict factor of it is admissible, then $x \rightarrow C$ is the column of reading $\widetilde{w}$ obtained from w by applying relation of type (R_3). In this case, the highest weight vertex of $B(w)$ is equal to $1 \cdots p\overline{p}$.

– If w is not a column word, then $x \rightarrow C$ is obtained by applying relations of type (R_1) or (R_2) to the final subword of length 3 of w. On the resulting word, one continues by applying relations of type (R_1) or (R_2) to the maximal overlapping subword of length 3 to the left and this procedure is repeated until the first subword of length 3 has been operated. The result is the reading of a symplectic tableau consisting of a column C' of height p and a column $\boxed{x'}$, where x' is an element of $\mathcal{C}_n$. Then

$$x \rightarrow C = C'\boxed{x'} = P(w).$$

In this case, the highest weight vertex of $B(w)$ is equal to $1 \cdots p1$.

3.3.2. Example. 1. Suppose $w(C) = 36\overline{6}\,\overline{4}$ and $x = \overline{3}$, then

$$\overline{3} \rightarrow \begin{array}{|c|}\hline 3 \\ \hline 6 \\ \hline \overline{6} \\ \hline \overline{4} \\ \hline \end{array} = \begin{array}{|c|}\hline 3 \\ \hline 6 \\ \hline \overline{6} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array}.$$

2. Suppose $w(C) = 14\bar{4}\,\bar{3}$ and $x = \bar{2}$, the word $14\bar{4}\,\bar{3}\,\bar{2}$ is a nonadmissible column word such that each strict subword of which is an admissible column word, then we obtain by applying relation of type (R_3),

$$\bar{2} \to \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|}\hline 1 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}.$$

3. Suppose $w(C) = 14\bar{4}\,\bar{3}$ and $x = 2$, then the word $14\bar{4}\,\bar{3}2$ is not a column word. By applying relations of type (R_1) or (R_2), we obtain:

$$14\boldsymbol{\bar{4}\,\bar{3}2} \equiv 1\boldsymbol{4\bar{4}2}\bar{3} \equiv \boldsymbol{142}\bar{4}\,\bar{3} \equiv 412\bar{4}\,\bar{3}.$$

Then

$$2 \to \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \\ \cline{1-1} \bar{4} & \\ \cline{1-1} \bar{3} & \\ \cline{1-1} \end{array}.$$

3.3.3. Insertion of a letter in a symplectic tableau. Let $T = C_1 \cdots C_r$ be a symplectic tableau with admissible column C_i, for $i = 1, \cdots, r$, and x be a letter.

1. If $w(C_1)x$ is an admissible column word, then $x \to T$ is the tableau obtained by adding a box filled by x on the bottom of C_1.

2. If $w(C_1)x$ is a nonadmissible column word such that each strict factor of it is an admissible column word. Let $\widetilde{w(C_1)}x = y_1 \cdots y_s$ be the admissible column word obtained from $w(C_1)x$ by applying relation of type (R_3) and $\widehat{T} = C_2 \cdots C_r$ be the tableau obtained from T after eliminating the first column C_1. Then $x \to T$ is obtained by inserting successively the elements of $\widetilde{w(C_1)}x$ in the tableau $\widehat{T}$. That is,
$$x \to T = y_s \to (y_{s-1} \to (\cdots y_1 \to \widehat{T})).$$
Moreover, the insertion of $y_1, \cdots, y_s$ in $\widehat{T}$ does not cause a new contraction.

3. If $w(C_1)x$ is not a column word, then
$$x \to C_1 = \begin{array}{|c|c|}\hline C_1' & y \\ \hline \end{array},$$
where C_1' is an admissible column of height $h(C_1)$ and y a letter. Then
$$x \to T = C_1'(y \to C_2 \cdots C_r),$$
that is, $x \to T$ is the juxtaposition of C_1' with the tableau obtained by inserting y in the tableau $C_2 \cdots C_r$.

3.3.4. Example. Consider a symplectic tableau

$$
T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
$$

and a letter $x = 1$. Let us compute $x \to T_1$. First, we begin inserting x in the first column C_1 of T_1. The word 1231 is not a column word, then by applying at each step (R_1) or (R_2), we obtain :

$$
1\mathbf{231} \equiv \mathbf{121}3 \equiv 1123,
$$

so

$$
1 \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.
$$

Then $1 \to T_1 = C_1'(1 \to T_1')$, where

$$
C_1' = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \text{ and } T_1' = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.
$$

Similarly, we have $2\mathbf{\bar{3}\,\bar{2}1} \equiv \mathbf{2\bar{3}1}\bar{2} \equiv 21\bar{3}\,\bar{2}$, then

$$
1 \to \begin{array}{|c|}\hline 2 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.
$$

So $1 \to T_1 = C_1'C_2'(2 \to \boxed{3})$, where

$$
C_2' = \begin{array}{|c|}\hline 1 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array}.
$$

Finally, we have $32 \equiv 32$, then

$$
2 \to \boxed{3} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}.
$$

Hence,

$$
1 \to T_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & \bar{3} & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \bar{2} & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}.
$$

3.3.5. Example. Consider a symplectic tableau

$$
T_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \bar{3} & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}.
$$

and a letter $x = \bar{3}$. Let us compute $x \to T_2$. First, we begin inserting $x = \bar{3}$ in the first column C_1 of T_2. The word $123\bar{3}$ is a nonadmissible column, that each strict factor is an admissible column word, we have by applying (R_3),

$$123\bar{3} \equiv 12,$$

then

$$\widetilde{C_1} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \text{ and } \widehat{T_2} = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 3&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array}.$$

So we have to insert the elements of the column $\widetilde{C_1}$ in the tableau $\widehat{T_2}$.

First, one inserts 1 :

$$13\mathbf{\bar{3}1} \equiv \mathbf{131}\bar{3} \equiv 113\bar{3},$$

then

$$1 \to \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline \bar{3}\\ \hline\end{array} = \begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1} \bar{3}\\ \cline{1-1}\end{array}.$$

We have $231 \equiv 213$, then

$$1 \to \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array} = \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}.$$

And

$$2 \to \begin{array}{|c|}\hline 3\\ \hline\end{array} = \begin{array}{|c|c|}\hline 2&3\\ \hline\end{array}.$$

Hence

$$1 \to \widehat{T_2} = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 3&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array} = \widehat{T_2}'.$$

Secondly, one inserts 2 in the tableau $\widehat{T_2}'$:

we have $13\mathbf{\bar{3}2} \equiv \mathbf{132}\bar{3} \equiv 312\bar{3}$, then

$$2 \to \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline \bar{3}\\ \hline\end{array} = \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1} \bar{3}\\ \cline{1-1}\end{array}.$$

We have $133 \equiv 313$, then

$$3 \to \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} = \begin{array}{|c|c|}\hline 1&3\\ \hline 3\\ \cline{1-1}\end{array}.$$

We have $23 \equiv 23$, then

$$3 \to \begin{array}{|c|}\hline 2\\ \hline\end{array} = \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}.$$

Hence,

$$2 \to \widehat{T_2}' = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3&3\\ \cline{1-3} \bar{3}\\ \cline{1-1}\end{array} = \bar{3} \to T_2.$$

3.3.6. Remark. Consider a word w on $\mathcal{C}_n^*$. The symplectic tableau $P(w)$ can be computed by starting with the empty word, which is a valid tableau, and iteratively applying the insertion schemes described above. Notice that when w is the reading of a symplectic tableau T, $P(w) = T$.

Let u and v be the readings of two admissible columns U and V respectively. As we have seen in the subsection 3.1, $U \succeq V$ means that the column U appears to the right of V in a symplectic tableau. Note that $U \not\succeq V$ means that the word uv is the reading of a nonsymplectic tableau.

3.3.7. Lemma. *Let u and v be the readings of two admissible columns U and V respectively, such that $U \not\succeq V$. The symplectic tableau $P(uv)$ consists of at most two columns.*

Proof. Let $u = x_1 \cdots x_p$ and $v = y_1 \cdots y_q$ be respectively the readings of two admissible columns U and V of height p and q, such that $U \not\succeq V$. We begin inserting the first element y_1 of v in the column U. The shape of $P(uy_1)$ depends of the connected component containing uy_1. The highest weight vertex of this component may be written $u^0y_1^0$. By Lemma 2.2.2, u^0 is of highest weight and $\varepsilon_i(y_1^0) \leq \varphi_i(u^0)$, for any $i = 1, \cdots, n$. Then we obtain the following cases :

(a) $u^0y_1^0 = 1 \cdots p(p+1)$. In this case, uy_1 is an admissible column word and wt($y_1^0 = p+1$) $=\Lambda_{p+1} - \Lambda_p$. Then during the insertion of the letter y_1 in the column U, this column of height p corresponding to the weight Λ_p is transformed into a column of height $p+1$ corresponding to the weight Λ_{p+1}. Its reading is uy_1. After one continues inserting the others elements $y_2, \cdots, y_q$ of the column word v. We know by the definition of an admissible column that every element of this column is strictly larger than its preceding, then we have:

First, suppose that $y_i^0 = p+i$, for $i = 2, \cdots, q$. Then $\mathrm{wt}(y_i^0) = \Lambda_{p+i} - \Lambda_{p+i-1}$ and during the insertion of y_i in the column of reading $uy_1 \cdots y_{i-1}$, this column of height $p+i-1$ is turned into the column of reading $uy_1 \cdots y_i$ and of height $p+i$. Thus uv is an admissible column word and $P(uv)$ consists of one column uv.

Secondly, suppose that there exists an element y_k of the column word v such that $uy_1 \cdots y_{k-1}y_k$ is a nonadmissible column word whose each strict factor is an admissible column word, then $y_k^0 = \overline{p+k-1}$ and $\mathrm{wt}(y_k^0) = \Lambda_{p+k-2} - \Lambda_{p+k-1}$, than during the insertion of y_k in the admissible column of reading $uy_1 \cdots y_{k-1}$, this column of height $p+k-1$ is transformed into a column of height $p+k-2$. After one continues inserting the remaining elements of v, then one adds those letters in distincts rows in the considered column or one removes some letters from distincts rows of the same column.

Hence, in this case $P(uv)$ consists of one column.

(b) $u^0y_1^0 = 1 \cdots p\overline{p}$. In this case, uy_1 is a nonadmissible column word such that each strict factor is an admissible column word. We have $\mathrm{wt}(y_1^0) = \Lambda_{p-1} - \Lambda_p$, then during the insertion of y_1 in the admissible column U, this column of height p is turned into a column of height $p-1$. Since the elements of the column V are strictly increasing , one can prove by similar arguments of the case (a), that during the computation of $P((uy_1)y_2 \cdots y_q)$, one adds a number of boxes of the considered column in distincts rows and one removes some boxes from distincts rows of the same column. Hence, we have also in this case, that $P(uv)$ consists of one column.

(c) $u^0y_1^0 = 1 \cdots p1$. In this case, uy_1 is not a column word, then during the insertion of y_1 in the admissible column U, an element appears in a second column. After, one inserts the next element y_2

of the column V in $P(uy_1)$, the highest weight of the connected component containing $w(P(uy_1))y_2$ may be written $w(P(uy_1)^0)y_2^0$, where $w(P(uy_1)^0)$ is of highest weight and by Lemma 2.2.2, we have:

(i) $y_2^0 = i$ (with $i = p+1$ or $i = 2$), then its weight is equal to $\Lambda_i - \Lambda_{i-1}$, than during the insertion $y_2 \to P(uy_1)$ a column of height $i-1$ is turned into a column of height i. Then one adds a box in the first column or in the second column of $P(uy_1)$.

(ii) $y_2^0 = \overline{p}$, then its weight is equal to $\Lambda_{p-1} - \Lambda_p$, then during the insertion $y_2 \to P(uy_1)$, the first column of height p is turned into a column of height $p-1$.

After we continue inserting the remaining letters of v, and since every element is strictly larger than its preceding, one adds boxes in distincts rows in the first or in the second column and similarly one removes boxes from distinct rows of the considered symplectic tableau. Hence, $P(uv)$ consists of two columns.

□

3.3.8. Lemma. *Let* u *and* v *be the readings of two admissible columns* U *and* V *respectively, such that* $U \not\succeq V$. *Then the column* U *contains more elements than* W, *the right column of* $P(uv)$.

Proof. Let $u = x_1 \cdots x_p$ and $v = y_1 \cdots y_q$ be respectively the readings of two admissible columns U and V of height p and q, such that $U \not\succeq V$. Let w and w' be respectively the readings of the right and left column W and W' of $P(uv)$. If the height of U is greater than the height of V, then in all cases we have $h(W) < p$. Suppose now that $q \geq p$ and the columns U and V contain only unbarred letters. Suppose that during the computation of $P(uv)$, we only add boxes by applying relations of type (R_1). If $h(W) = p$, then during inserting the first p elements of V, p boxes are added in the second column and they are all filled by elements of U. Since the number of added boxes is equal to the height of U, $w(P(uv)) = uv$. Then $U \succeq V$ which yields a contradiction. Hence, $h(W) < p$.

Suppose now that during the computation of $P(uv)$, we only add boxes by applying relations of type (R_1) or (R_2). By the definition of $P(uv)$ we have $w(P(uv)) = ww' \equiv uv$. Then the words uv and ww' occur at the same place in their isomorphic connected components $B(uv)$ and $B(ww')$ of the crystal G_n. Note that all the vertices in a connected component are the readings of tableaux of same shape. Let $(uv)^0$ and $(ww')^0$ be respectively the highest weight vertices of $B(uv)$ and $B(ww')$. The word $(ww')^0$ is the reading of a tableau that all its elements are unbarred letters, then $(uv)^0$ and $(ww')^0$ are related by relations of type (R_1). Hence, as we have seen above, the height of the second column of $P((uv)^0)$ is strictly less than p. Since $(ww')^0$ and ww' are the readings of two symplectic tableaux of same shape, the length of w is stricly less than p.

Suppose that during the insertion of the first k elements of v, for $k \leq p-1$, into the column U, we add k boxes in a second column. Then

$$P(uy_1 \cdots y_k) = \boxed{\begin{array}{c|c} C_1 & C_2 \end{array}},$$

where C_1 contains p elements and C_2 contains the k added boxes. After we insert y_{k+1} in the column C_1. Suppose that $w(C_1)y_{k+1}$ is a nonadmissible column word such that all of its proper factors are admissible. Let $\widetilde{w(C_1)y_{k+1}}$ be the column word obtained from $w(C_1)y_{k+1}$ after applying relation of type (R_3). Then we insert the elements of $w\widetilde{(C_1)y_{k+1}}$ in the column C_2. This insertion does not cause a new contraction. Then if we obtained two columns, the heigth of the right one is strictly less than the heigth of C_2 which is strictly less than p. After we continue inserting the remaining elements of v, and the heigth of the right column of the final tableau is strictly less than p. $\square$

4. Convergent presentation of plactic monoid for type C

We consider a presentation of the symplectic plactic monoid $\mathbf{Pl}(C)$, by the 2-polygraph $\Sigma^{Sp(n)}$, whose set of 1-cells is $\mathcal{C}_n$ and whose 2-cells correspond to the relations (R_1), (R_2) and (R_3) oriented with respect to the reverse deglex order, that is

$$\begin{aligned}
\Sigma_2^{Sp(n)} = & \left\{ xzy \overset{\kappa_{x,y,z}}{\Longrightarrow} zxy \;\middle|\; x < y \leq z \text{ and } z \neq \overline{x} \right\} \\
& \cup \left\{ yxz \overset{\kappa'_{x,y,z}}{\Longrightarrow} yzx \;\middle|\; x \leq y < z \text{ and } z \neq \overline{x} \right\} \\
& \cup \left\{ yx\overline{x} \overset{\xi_{x,y,\overline{x}}}{\Longrightarrow} y\overline{(x-1)}(x-1) \;\middle|\; x \leq y \leq \overline{x} \text{ and } 1 < x \leq n \right\} \\
& \cup \left\{ x\overline{x}y \overset{\xi'_{x,y,\overline{x}}}{\Longrightarrow} \overline{(x-1)}(x-1)y \;\middle|\; x \leq y \leq \overline{x} \text{ and } 1 < x \leq n \right\} \\
& \cup \left\{ w \overset{\zeta_w}{\Longrightarrow} \widetilde{w} \;\middle|\; w \text{ and } \widetilde{w} \text{ satisfy the conditions of the relation } (R_3) \right\}.
\end{aligned}$$

The order being monomial, the 2-polygraph $\Sigma^{Sp(n)}$ is terminating.

4.0.9. Theorem. *For $n \geq 4$, there is no finite completion of the 2-polygraph $\Sigma^{Sp(n)}$ compatible with the reverse deglex order.*

Proof. We know that for $n = 4$, the Knuth presentation of the plactic monoid for type A doesn't admit a finite completion compatible with the reverse deglex order because during the completion one adds an infinity of 2-cells of the form $232^i124 \Longrightarrow 2342^i12$, for $i > 1$. Since the 2-polygraph $\Sigma^{Sp(n)}$ contains the Knuth relations for type A, it doesn't also admit a finite completion compatible with the reverse deglex order. $\square$

In order to give a finite convergent presentation of the symplectic plactic monoid $\mathbf{Pl}(C)$, one introduces the admissible column generators. The set of generators is

$$\Gamma_1 = \left\{ c_u \;\middle|\; u \text{ is an admissible column word of } \mathcal{C}_n^* \right\},$$

where each symbol c_u represents the element u of $\mathbf{Pl}(C)$. We know that $\mathcal{C}_n$ consists of columns of one element, then the set Γ_1 also generates $\mathbf{Pl}(C)$.

We consider the two following sets of 2-cells, the 2-cells corresponding to the relations $(R_1), (R_2)$

and (R_3), that is,

$$
\begin{aligned}
\Gamma_2^{\mathrm{Sp(n)}} = & \left\{ c_x c_z c_y \overset{c_{\kappa_{x,y,z}}}{\Longrightarrow} c_z c_x c_y \;\middle|\; x < y \leq z \text{ and } z \neq \overline{x} \right\} \\
& \cup \left\{ c_y c_x c_z \overset{c_{\kappa'_{x,y,z}}}{\Longrightarrow} c_y c_z c_x \;\middle|\; x \leq y < z \text{ and } z \neq \overline{x} \right\} \\
& \cup \left\{ c_y c_x c_{\overline{x}} \overset{c_{\xi_{x,y,\overline{x}}}}{\Longrightarrow} c_y c_{\overline{(x-1)}} c_{(x-1)} \;\middle|\; x \leq y \leq \overline{x} \text{ and } 1 < x \leq n \right\} \\
& \cup \left\{ c_x c_{\overline{x}} c_y \overset{c_{\xi'_{x,y,\overline{x}}}}{\Longrightarrow} c_{\overline{(x-1)}} c_{(x-1)} c_y \;\middle|\; x \leq y \leq \overline{x} \text{ and } 1 < x \leq n \right\} \\
& \cup \left\{ c_{x_1} \cdots c_{x_{|w|}} \overset{c_{\zeta_w}}{\Longrightarrow} c_{\widetilde{x}_1} \cdots c_{\widetilde{x}_{|\widetilde{w}|}} \;\middle|\; w \text{ and } \widetilde{w} \text{ verify the conditions of the relation} (R_3) \right\}.
\end{aligned}
$$

where $w = x_1 \cdots x_{|w|}$ and $\widetilde{w} = \widetilde{x}_1 \cdots \widetilde{x}_{|\widetilde{w}|}$, and the 2-cells corresponding to the defining relations for the extra column generators c_u, where $|u| \geq 2$.

$$
\Gamma_2^{c(n)} = \left\{ c_{y_1} \cdots c_{y_k} \overset{\gamma_{y_1,\dots,y_k}}{\Longrightarrow} c_{y_1 \cdots y_k} \;\middle|\; y_1 \cdots y_k \text{ is an admissible column word of } \mathcal{C}_n^* \right\}.
$$

The monoid $\mathbf{Pl}(C)$ is presented by the 2-polygraph $\Gamma^{(n)} = (\Gamma_1, \Gamma_2^{(n)})$, with $\Gamma_2^{(n)} = \Gamma_2^{\mathrm{Sp(n)}} \cup \Gamma_2^{c(n)}$.

Let u and v be respectively the readings of two admissible columns U and V. Suppose that $U \npreceq V$, let us define a 2-cell

$$
c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}
$$

where the words w and w' are respectively the readings of the right and left columns W and W' of $P(uv)$ if this symplectic tableau consists of two columns. Moreover, if $P(uv)$ consists of one column W, then w' is the empty word.

Define

$$
\Omega_2 = \left\{ c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'} \;\middle|\; u \text{ and } v \text{ are admissible columns words of } \mathcal{C}_n^* \text{ and } U \npreceq V \right\}.
$$

The 2-polygraph $\Sigma^{\mathrm{acol(n)}} = (\Gamma_1, \Omega_2)$ is called the *admissible column presentation* .

Let us define an order on Γ_1^*. First, choose an order $\sqsubset$ on Γ_1 such that $c_u \sqsubset c_v$ if $|u| < |v|$. Secondly, consider the order $\prec$ on Γ_1^*, defined as follows. We have

$$
\begin{aligned}
& c_{u_1} c_{u_2} \cdots c_{u_m} \prec c_{v_1} c_{v_2} \cdots c_{v_n} \\
& \text{if } \left[(m < n) \text{ or } \left(m = n \text{ and there exists } i \text{ such that: } c_{u_i} \sqsubset c_{v_i} \text{ and } \forall j < i,\ c_{u_j} = c_{v_j} \right) \right]
\end{aligned}
$$

where c_{u_i} and c_{v_j} are elements of Γ_1, for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Thus $\prec$ is a well-ordering on Γ_1^*.

4.0.10. Lemma. *The 2-polygraph $\Sigma^{acol(n)}$ is finite.*

Proof. The symplectic tableau $P(u)$ is unique for each word u, then for two admissible columns words u and v of $\mathcal{C}_n^*$, the 2-cells of the form $\alpha_{u,v}$ are uniquely determined. In addition, Γ_1 is finite thanks to the fact that the admissible columns words of $\mathcal{C}_n^*$ have length at most n. $\square$

The following lemma shows that the plactic monoid $\mathbf{Pl}(C)$ is presented by the 2-polygraph $\Sigma^{\mathrm{acol}(n)}$:

4.0.11. Lemma. *The 2-polygraphs $\Gamma^{(n)}$ and $\Sigma^{acol(n)}$ are tietze equivalent.*

Proof. Every relation in $\Gamma_2^{\mathrm{Sp}(n)}$ can be deduced from rules in Ω_2, indeed,

$$
\begin{tikzcd}
c_z c_x c_y \arrow[d, Rightarrow, "c_z\alpha_{x,y}"'] & c_x c_z c_y \arrow[l, Rightarrow, "c_{\kappa_{x,y,z}}"'] \arrow[d, Rightarrow, "\alpha_{x,z}c_y"] \\
c_z c_{xy} & c_{xz} c_y \arrow[l, Rightarrow, "\alpha_{xz,y}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
c_y c_z c_x \arrow[d, Rightarrow, "\alpha_{y,z}c_x"'] & c_y c_x c_z \arrow[l, Rightarrow, "c_{\kappa'_{x,y,z}}"'] \arrow[d, Rightarrow, "c_y\alpha_{x,z}"] \\
c_{yz} c_x \arrow[r, Rightarrow, "\alpha_{yz,x}"'] & c_y c_{xz}
\end{tikzcd}
$$

$$
\begin{tikzcd}
c_y c_{\overline{x-1}} c_{x-1} \arrow[d, Rightarrow, "\alpha_{y,\overline{x-1}}c_{x-1}"'] & c_y c_x c_{\overline{x}} \arrow[l, Rightarrow, "c_{\xi_{x,y,\overline{x}}}"'] \arrow[d, Rightarrow, "c_y\alpha_{x,\overline{x}}"] \\
c_{y\overline{x-1}} c_{x-1} \arrow[r, Rightarrow, "\alpha_{y\overline{x-1},x-1}"'] & c_y c_{x\overline{x}}
\end{tikzcd}
\qquad
\begin{tikzcd}
c_{\overline{x-1}} c_{x-1} c_y \arrow[d, Rightarrow, "c_{\overline{x-1}}\alpha_{x-1,y}"'] & c_x c_{\overline{x}} c_y \arrow[l, Rightarrow, "c_{\xi'_{x,y,\overline{x}}}"'] \arrow[d, Rightarrow, "\alpha_{x,\overline{x}}c_y"] \\
c_{\overline{x-1}} c_{(x-1)y} & c_{x\overline{x}} c_y \arrow[l, Rightarrow, "\alpha_{x\overline{x},y}"']
\end{tikzcd}
$$

Let $w = x_1 \cdots x_p \cdots x_q \cdots x_k$ be a nonadmissible column word of length k such that each strict factor of w is an admissible column word. Let $z = x_p$ be the lowest unbarred letter such that the pair $(z = x_p, \overline{z} = x_q)$ occurs in w and $N(z) = z + 1$. Then we have :

$$
\begin{tikzcd}
c_{x_1}\cdots c_{x_p}\cdots c_{x_q}\cdots c_{x_k} \arrow[rrr, Rightarrow, "c_{\zeta_w}"] \arrow[d, Rightarrow, "\alpha_{x_1,x_2}c_{x_3}\cdots c_{x_k}"'] & & & c_{x_1}\cdots\widehat{c_{x_p}}\cdots\widehat{c_{x_q}}\cdots c_{x_k} \arrow[d, Rightarrow, "\alpha_{x_1,x_2}c_{x_3}\cdots c_{x_k}"] \\
(\cdots) \arrow[d, Rightarrow, "\alpha_{x_1\cdots x_{q-2},x_{q-1}}c_{x_q}\cdots c_{x_k}"'] & & & (\cdots) \arrow[d, Rightarrow, "\alpha_{x_1\cdots\widehat{x_p}\cdots\widehat{x_q}\cdots x_{k-1},x_k}"] \\
c_{x_1\cdots x_{q-1}}c_{x_q}\cdots c_{x_k} \arrow[r, Rightarrow, "\alpha_1"] & c_{x_1\cdots\widehat{x_p}\cdots x_{q-1}\widehat{x_q}}c_{x_{q+1}}\cdots c_{x_k} \arrow[r, Rightarrow, "\alpha_2"] & (\cdots) \arrow[r, Rightarrow, "\alpha_3"] & c_{x_1\cdots\widehat{x_p}\cdots\widehat{x_q}\cdots x_k}
\end{tikzcd}
$$

where the symbol $\hat{x}$ means that x is removed, $\alpha_1 = \alpha_{x_1\cdots x_{q-1},x_q} c_{x_{q+1}} \cdots c_{x_k}$, $\alpha_2 = \alpha_{x_1\cdots\hat{x_p}\cdots x_{q-1}\hat{x_q},x_{q+1}} c_{x_{q+2}} \cdots c_{x_k}$ and $\alpha_3 = \alpha_{x_1\cdots\hat{x_p}\cdots\hat{x_q}\cdots x_{k-1},x_k}$.

In addition, any rules in $\Gamma_2^{c(n)}$ can be obtained using those in Ω_2 :

$$
\begin{tikzcd}
c_{u_1}\ldots c_{u_k} \arrow[rr, Rightarrow, "\gamma_{u_1,\ldots,u_k}"] \arrow[d, Rightarrow, "\alpha_{u_1,u_2}c_{u_3}\ldots c_{u_k}"'] & & c_{u_1\ldots u_k} \\
c_{u_1u_2}c_{u_3}\ldots c_{u_k} \arrow[r, Rightarrow, "\alpha_{u_1u_2,u_3}c_{u_4}\ldots c_{u_k}"] & (\cdots) \arrow[r, Rightarrow, "\alpha_{u_1\ldots u_{k-2},u_{k-1}}c_{u_k}"] & c_{u_1\ldots u_{k-1}}c_{u_k} \arrow[u, Rightarrow, "\alpha_{u_1\ldots u_{k-1},u_k}"']
\end{tikzcd}
$$

□

4.0.12. Remark. Every rule in Ω_2 holds in the symplectic plactic monoid $\mathbf{Pl}(C)$, indeed,

$$c_u c_v \equiv uv \equiv w(P(uv)) = ww' \equiv c_w c_{w'}.$$

4.0.13. Theorem. *The 2-polygraph $\Sigma^{acol(n)}$ is a finite convergent presentation of the symplectic plactic monoid* $\mathbf{Pl}(C)$.

Proof. By Lemma 4.0.10, the 2-polygraph $\Sigma^{\mathrm{acol(n)}}$ is finite. Let us show that it also convergent. First, the 2-polygraph $\Sigma^{\mathrm{acol(n)}}$ is terminating. Indeed, we have to prove that if $h \Longrightarrow h'$, then $h' \prec h$. One finds two cases :

- First case : let $h = pc_uc_vq$ and $h' = pc_wq$, with $p, q \in \Gamma_1^*$ and $c_u, c_v, c_w \in \Gamma_1$. One remarks that h' is shorter than h, then $h' \prec h$.

- Second case : let $h = pc_uc_vq$ and $h' = pc_wc_{w'}q$, with $p, q \in \Gamma_1^*$ and $c_u, c_v, c_w, c_{w'} \in \Gamma_1$, where w and w' are respectively the readings of the right and left columns of $P(uv)$. One remarks that h and $h^{'}$ have the same length. By Lemma 3.3.8 the length of u is strictly larger than the length of w, then $c_w \sqsubset c_u$. If we consider $i = |p| + 1$, then we have $c_{u_i} = c_w \sqsubset c_u = c_{v_i}$ and for all $j < i$, $c_{u_j} = c_{v_j}$. Then, $h' \prec h$. Since every application of a 2-cell of Ω_2 yields a $\prec$-preceding 1-cell, it follows that any sequence of rewriting using Ω_2 must terminate. Hence, the 2-polygraph $\Sigma^{\mathrm{acol(n)}}$ is terminating.

Secondly, the 2-polygraph $\Sigma^{\mathrm{acol(n)}}$ is confluent. Indeed, let $h \in \Gamma_1^*$ and h', h'' be two normal forms obtained from h. Since $\Sigma^{\mathrm{acol(n)}}$ is terminating, h' and h'' exist. We have to prove that $h = h'$. Suppose that $h' = c_{u_k} \cdots c_{u_1}$. Since h' is a normal form, the words $u_1, \cdots, u_k$ are respectively the readings of k admissible columns $U_1, \cdots, U_k$ of a symplectic tableau, i.e, $U_i \preceq U_{i+1}, \forall i$. Then $u_k \cdots u_1 = w(T')$, where T' is the unique symplectic tableau such that

$$w(T') = u_k \cdots u_1 \equiv h'.$$

Similarly, $h'' = c_{v_l} \cdots c_{v_1}$ is a normal form, then there exists a unique symplectic tableau T" such that

$$w(T'') = v_l \cdots v_1 \equiv h''.$$

Since $h \equiv h' \equiv h''$, we have by Theorem 3.2.3 that $T' = T''$. Then we have $k = l$ and $u_i = v_i, \forall i = 1, \cdots, k$. Thus $h' = h''$.

Hence, the 2-polygraph $\Sigma^{\mathrm{acol(n)}}$ is convergent. □

Since plactic monoids for type C admit finite convergent presentations, we obtain by Corollary 4.5.4 in [GM12] that they satisfy the homotopical finiteness condition FDT_∞. This result yields by Corollary 5.4.4 in [GM12] that plactic monoids for type C satisfy the homological finiteness property type FP_∞.

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